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COMMENT

Comment on 'Painlevé test and integrability of non-linear Klein–Fock–Gordon equations'

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Abstract. We show that the arguments of Doktorov and Sakovich against the validity of the Painlevé test are unfounded, being based on an incomplete singularity analysis.

Doctorov and Sakovich (1985) is based on a misconception of the so-called 'Painlevé analysis'. They consider part of the possible singular behaviour of the solution of an equation. Finding that *some* behaviour has the Painlevé property, they mistakenly jump to the conclusion that the equation passes the test. In fact, most of the equations they quote *fail* the test, because of a singular behaviour the authors ignored. In particular, equations (22), (25) and '(4) with a glance to (26)' *fail* to pass the test in general, contrary to the authors' assertion.

Most papers on singularity analysis deal with differential equations polynomial in the dependent variable φ where the coefficient of the highest derivative term is a constant. Therefore singularities occur only when φ diverges. The crucial point is that for equations like (4), where the coefficient $A(\varphi)$ of the highest derivative term is *not* a constant, singularities occur not only when φ diverges, but also when $A(\varphi)$ goes to zero, i.e. for φ going to zero in their case where A is a monomial in φ. Therefore one must check both that 'pole-like' singularities are just pure poles, but also that 'zero-like' points are regular zeros and not critical points.

The authors' claim that equation (25):

$$\psi_{xy}=\tilde{\xi}_1\,\mathrm{e}^{2\psi}+\tilde{\xi}_2\,\mathrm{e}^{-\psi}+\tilde{\xi}_3\,\mathrm{e}^{-2\psi}$$

passes the Painlevé test is self-contradictory. Their own calculations (equation (24)) prove that if $\tilde{\xi}_1$ is not zero, no term proportional to e^{ψ} is allowed. Change ψ to $-\psi$ (and certainly any reasonable definition of the Painlevé property should be invariant under this change), which precisely corresponds to considering 'zeros' of $\varphi=\mathrm{e}^{\psi}$ instead of 'poles'. One similarly concludes that if $\tilde{\xi}_3$ is not zero, $\tilde{\xi}_2$ must vanish in order for the Painlevé property to hold. For the same reason, (25) contradicts (22):

$$\psi_{xy}=\xi_1\,\mathrm{e}^{\psi}+\xi_2\,\mathrm{e}^{-\psi}+\xi_3\,\mathrm{e}^{-2\psi}.$$

When $\tilde{\xi}_1\,\mathrm{e}^{2\psi}$ is present in (25) no term proportional to e^{ψ} is allowed. Change ψ to $-\psi$, take $\tilde{\xi}_3$ to be zero, which now allows a non-zero $\tilde{\xi}_2$. One recovers (22) with $\xi_1=-\tilde{\xi}_2$, $\xi_3=-\tilde{\xi}_1$, but where ξ_2 must be zero!

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As a result, the only equations of this form that pass the Painlevé test are the three known integrable equations: Liouville, sine-Gordon and Mikhailov-Dodd-Bullough. This has been proven by Clarkson *et al* (1986).

As for equations (4)-(26), the mistake is quite obvious. The authors consider pole-like singularities of $\varphi = 1/\psi$ and find them to be pure poles, as should be expected. Indeed, these are zeros of ψ (satisfying (27)) which are manifestly regular points. What are interesting are the pole-like singularities of equation (27), i.e. zero-like singularities of equations (4)-(26). The condition for these singularities to be in fact regular zeros (i.e. pure poles of ψ) are highly non-trivial. Some of these equations (very special) *may* have the Painlevé property and will presumably be integrable. In general, however, this is not the case. Let us consider the simplest non-trivial equation of the type (4)-(26), namely

$$\psi_{xy} = \psi^3 \tag{1}$$

which is well known not to be integrable. This is consistent with the fact that it does not pass the Painlevé test in the sense of Weiss-Kruskal (Weiss *et al* 1983, Kruskal 1980 private communication, Jimbo *et al* 1982) as follows.

Let us look for a pole-like expansion of the form, where g is a free function of y:

$$\psi = \sum_{n=0}^{\infty} a_n(y)[x - g(y)]^{n-1}.$$

We find a resonance at $n = 4$. Order by order one obtains

$$a_0^2(y) = -2\partial g(y)/\partial y$$
$$a_1(y) = -(\partial a_0(y)/\partial y)/3a_0^2(y)$$
$$a_2(y) = -a_1^2(y)/a_0(y)$$
$$a_3(y) = [5a_1^3(y)/2 - a_1(y)\partial a_1(y)/\partial y]/a_0^2(y).$$

At order four the resonance condition is

$$\frac{\partial}{\partial y}[a_0(y)a_3(y)] = 0$$

which is *not* identically satisfied but is an equation for $g(y)$. In the sense of the Weiss-Kruskal conjecture this precisely means that equation (1) *fails* the test as is expected since it is known to be non-integrable.

Incidentally, another objection of Doktorov and Sakovich to the Weiss algorithm (where ω and ε_r depend on N variables when only two function of $N-1$ variables are needed) has been taken care of long ago by the Kruskal modification of the Weiss algorithm.

Even though some other points stressed by the authors, like the role of essential singularities, may be relevant, it remains that their paper is based on a severe misconception. The connection between integrability and the Painlevé property is not yet clearly established and some subtle and delicate problems remain. However, the reliability of the Painlevé method will only be established (or challenged) through complete singularity analysis.

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